

Description of the characters and factor representations of the infinite symmetric inverse semigroup*.

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Introduction

In this paper, we describe the characters of the infinite symmetric semigroup. The main result establishes a link between the representation theory of the finite symmetric semigroups developed by Munn [13], [14], Solomon [17], Halverson [12], Vagner [1], Preston [16], and Popova [11] on the one hand, and the representation theory of locally finite groups (in particular, the infinite symmetric group) and locally semisimple algebras developed in the papers by Thoma [18], Vershik and Kerov [4]–[6], [20] on the other hand. The below analysis of the Bratteli diagram for the infinite symmetric semigroup reminds the analogous analysis in the more complicated case of describing the characters of the Brauer–Weyl algebras [7]. The symmetric semigroup appeared not only in the literature on the theory of semigroups and their representations, but also in connection with the representation theory of the infinite symmetric group [15] and the definition of the braid semigroup [19]; q -analogs of the symmetric semigroup were also considered [12]. Apparently, the definition of the infinite symmetric semigroup given in this paper, as well as problems related to representations of this semigroup, have not yet been discussed in the literature.

Consider the set of all one-to-one *partial* transformations of the set $\{1, \dots, n\}$, i.e., one-to-one maps from a subset of $\{1, \dots, n\}$ to a subset (possibly, different from the first one) of $\{1, \dots, n\}$. We define the product of such maps as their composition where it is defined. Thus we obtain a semigroup with a zero (the map with the empty domain of definition), which is usually called the *symmetric inverse semigroup*; denote it by R_n (there are also other notations, see [9], [17]).

Obviously, the symmetric group S_n is a subgroup of the semigroup R_n : $S_n \subset R_n$. Further, R_n can be presented as the semigroup of all 0-1 matrices with at most one 1 in each row and each column equipped with matrix multiplication. This realization is similar to the natural representation of the symmetric group. The matrices of this form are in a one-to-one correspondence with all possible placements of nonattacking rooks on the $n \times n$ chessboard, that is why Solomon called this monoid (the semigroup with a zero) the *rook monoid*.

The following properties of inverse semigroups and, in particular, the symmetric inverse semigroup are of great importance (see the Appendix).

(1) the complex semigroup algebra of every finite inverse semigroup is semisimple ([10], [14]);

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(2) every finite inverse semigroup can be isomorphically embedded into a symmetric inverse semigroup ([1], [16]);

(3) the class of finite inverse semigroups generates exactly the class of involutive semisimple bialgebras [2].

The following result, which describes the characters of a finite inverse semigroup, was essentially discovered by several authors; its combinatorial and dynamical characterization is given in [12].

The set of irreducible representations (and, consequently, the set of irreducible characters) of the symmetric semigroup R_n is indexed by the set of all Young diagrams with at most n cells. The branching of representations in terms of diagrams looks as follows: when passing from an irreducible representation of R_n to representations of R_{n+1} , the corresponding Young diagram either does not change, or obtains one new cell (grows).

The infinite symmetric group S_∞ is the countable group of all finitary (i.e., nonidentity only on a finite subset) one-to-one transformations of a countable set. In the same way one can define the *infinite symmetric inverse semigroup*¹ R_∞ as the set of partial one-to-one transformations of a countable set that are nonidentity only on a finite subset.² The group S_∞ is the inductive limit of the chain S_n , $n = 1, 2, \dots$, with the natural embeddings of groups. In the same way, the semigroups R_n , $n = 1, 2, \dots$, form a chain with respect to the natural monomorphisms of semigroups³ $R_0 \subset R_1 \subset \dots \subset R_n \subset \dots$, and its inductive limit is the infinite inverse symmetric semigroup. The connection between the Bratteli diagram of the infinite symmetric group (the Young graph) and that of the infinite symmetric inverse semigroup leads naturally to introducing a new operation on graphs, which associates with every Bratteli diagram its “slow” version. (Cf. the notion of the “pascalization” of a graph introduced in [7].)

Our results rely on the well-developed representation theory of the infinite symmetric group S_∞ and, to some extent, generalize it. Recall that the list of characters of the infinite symmetric group was found by Thoma [18]. The new proof of Thoma’s theorem suggested by Vershik and Kerov [4] was based on approximation of characters of the infinite symmetric group by characters of finite symmetric groups and used the combinatorics of Young diagrams, which, as is well known, parameterize the irreducible complex representations of the finite symmetric groups. The parameters of indecomposable characters in the exposition of [4] are interpreted as the frequencies of the rows and columns of a sequence of growing Young diagrams. *The main result of this paper is that the list of parameters for the characters of the infinite symmetric group is obtained from the list of Thoma parameters by adding a new number from the interval $[0, 1]$.* The meaning of this new parameter is as follows. The irreducible representations of the finite symmetric semigroup R_n are also parameterized by Young diagrams, but with an arbitrary number of cells k not exceeding n ; thus, apart from the limiting frequencies of rows and columns, a sequence of growing diagrams has another parameter: the limit of the ratio k/n , which is the relative velocity with which the corresponding path

¹Usually we omit the word “inverse” and speak about the (infinite) symmetric semigroup.

²Thus the infinite symmetric inverse semigroup does not contain the zero map, since every element must be identity on the complement of a finite set.

³Under the monomorphism $R_n \subset R_{n+1}$, the zero of R_n is mapped not to a zero, but to a certain projection; more exactly, to the generator $p_n \in R_{n+1}$, see Theorem 1.7.

passes through the levels of the branching graph; or, in other words, the deceleration of the rate of approximation of a character of the infinite semigroup by characters of finite semigroups.

The description of the characters allows us to construct a realization of the corresponding representations. They live in the same space as the corresponding representations of the infinite symmetric group. More exactly, the space of the representation is constructed in exactly the same way as in the model of factor representations of the infinite symmetric group suggested in [5], but with the extended list of parameters, see Theorem 2.16.

In the first section, we give the necessary background on the representation theory of the finite symmetric inverse semigroups. The second section is devoted to the representation theory of the infinite symmetric semigroup R_∞ and contains our main results. In Appendix we collect general information about finite inverse semigroups and some new facts about their semigroup algebras regarded as Hopf algebras.

1. The representation theory of the finite symmetric inverse semigroups

1.1. The semisimplicity of the semigroup algebra $\mathbb{C}[R_n]$. The complete list of irreducible representations. We define the *rank* of a map $a \in R_n$ as the number of elements on which this map is not defined. Each of the sets $A_r = \{a \in R_n \mid \text{the rank of } a \text{ is at least } r\}$ for $0 \leq r \leq n$ is an ideal of the semigroup R_n . The chain of ideals

$$R_n = A_0 \supset A_1 \supset \cdots \supset A_n$$

is a principal series of the semigroup R_n , i.e., there is no ideal lying strictly between A_r and A_{r+1} , see Theorem 1.1.

Denote by $\mathbb{C}[S_n]$ the complex group algebra of the symmetric group S_n . This algebra, as well as the group algebra of every finite group, is semisimple, since in it there exists an invariant inner product.

The complex semigroup algebra of an inverse group is always semisimple too, as follows from the general Theorem 3.3. An explicit decomposition of the algebra $\mathbb{C}[R_n]$ into matrix components was suggested by Munn [13].

Theorem 1.1 (Munn). *The algebra $\mathbb{C}[R_n]$ is semisimple and has the form*

$$\mathbb{C}[R_n] \cong \bigoplus_{r=0}^n M_{\binom{n}{r}}(\mathbb{C}[S_r]).$$

Here $M_l(A)$ is the algebra of matrices of order l over an algebra A . A description of the representations of the algebra $\mathbb{C}[R_n]$ is given by the following theorem.

Theorem 1.2 (Munn). *Let S be a semigroup isomorphic to the semigroup $M_n(G)$ of $n \times n$ matrices with elements from a group G . Let F be a field whose characteristic is equal to zero or is a prime not dividing the order of G . Let $\{\gamma_p\}_{p=1}^k$ be the complete list of nonequivalent irreducible representations of the group G over F . Denote by γ'_p the map given by the formula*

$$\gamma'_p(\{x_{ij}\}) = \{\gamma_p(x_{ij})\}$$

for every matrix $\{x_{ij}\} \in S = M_n(G)$. Then $\{\gamma'_p\}_{p=1}^k$ is the complete list of nonequivalent irreducible representations of the semigroup S over F .

Denote by \mathcal{P}_r the set of all partitions of a positive integer r . It follows from the previous theorem that the set of irreducible representations of the semigroup R_n can be naturally indexed by the set $\bigcup_{r=0}^n \mathcal{P}_r$.

Remark 1.3. As can be seen from the form of irreducible representations of the semigroup R_n described above, each such representation is an extension of a uniquely defined induced representation of the group S_n . More exactly, for the irreducible representation of R_n corresponding to a partition $\lambda \in \mathcal{P}_r$, consider the representation of the subgroup $S_r \times S_{n-r} \subset S_n$ in which the action of S_r corresponds to the partition λ and S_{n-r} acts trivially. The corresponding induced representation of S_n can be extended to the original irreducible representation of R_n . (This was also observed in [15].)

Remark 1.4. On the semigroup algebra $\mathbb{C}[R_n]$ of the symmetric semigroup, as well as on the group algebra $\mathbb{C}[S_n]$ of the symmetric group, there is an involution, which, in particular, sends every irreducible representation π to the representation $\text{sgn}\pi$. It corresponds to the natural involution on the Young graph and, consequently, of the slow Young graph (for the definition, see Section 2.1) that sends a diagram to its reflection in the diagonal. However, it is not an involution of the group S_n or the semigroup R_n .

1.2. A formula for the characters of the finite symmetric semigroup.

Munn [13] also found a formula that expresses the characters of the symmetric inverse semigroup in terms of characters of the symmetric groups. In order to state the corresponding theorem, for every subset $K \subset \{1, \dots, n\}$, $|K| = r$, fix an arbitrary partial bijection $\mu_K: K \mapsto \{1, \dots, r\}$. By $\mu_K^-: \{1, \dots, r\} \mapsto K$ we denote the map inverse to μ_K on K ; thus $\mu_K^- \circ \mu_K$ is the identity map on the set K .

Theorem 1.5 (Munn). *Let χ^* be the character of the irreducible representation of the semigroup R_n corresponding to a partition $\lambda \in \mathcal{P}_r$, $1 \leq r \leq n$. Let χ be the corresponding character of the symmetric group S_r . Then for every element $\sigma \in R_n$,*

$$\chi^*(\sigma) = \sum \chi(\mu_K \sigma \mu_K^-),$$

where the sum is taken over all subsets K of the domain of definition of σ such that $|K| = r$ and $K\sigma = K$.

1.3. Presentations of the semigroup R_n by generators and relations.

We are interested in families of generators of the semigroups $\{R_n\}_{n=0}^\infty$ that increase under the embeddings $R_n \subset R_{n+1}$. This condition is satisfied for the generators suggested by Popova [11] and those suggested by Halverson [12]. In Halverson's paper, the generators and relations are described for a q -analog of the symmetric inverse semigroup. Below we present the particular case of his result for $q = 1$.

Let σ_i , $1 \leq i < n$, be the Coxeter generators of the group S_n . By $p_i \in R_n$, $1 \leq i \leq n$, we denote the following maps: $p_i(j)$ is not defined if $j \leq i$, and $p_i(j) = j$ if $j > i$.

Theorem 1.6 (Popova). *The semigroup R_n is generated by the elements $\sigma_1, \dots, \sigma_{n-1}, p_1$ with the following relations:*

- (1) the Coxeter relations for the group S_n ;
- (2) $\sigma_2 p_1 \sigma_2 = \sigma_2 \sigma_3 \cdots \sigma_{n-1} p_1 \sigma_2 \sigma_3 \cdots \sigma_{n-1} = p_1 = p_1^2$;
- (3) $(p_1 \sigma_1)^2 = p_1 \sigma_1 p_1 = (\sigma_1 p_1)^2$.

Theorem 1.7 (Halverson). *The semigroup R_n is generated by the elements $\sigma_1, \dots, \sigma_{n-1}, p_1, \dots, p_n$ with the following relations:*

- (1) *the Coxeter relations for the group S_n ;*
- (2) $\sigma_i p_j = p_j \sigma_i = p_j$ for $1 \leq i < j \leq n$;
- (3) $\sigma_i p_j = p_j \sigma_i$ for $1 \leq j < i \leq n-1$;
- (4) $p_i^2 = p_i$ for $1 \leq i \leq n$;
- (5) $p_{i+1} = p_i \sigma_i p_i$ for $1 \leq i \leq n-1$.

An interesting presentation of the semigroup R_n by generators and relations was suggested by Solomon [17]: in addition to the Coxeter generators of the group S_n , he considers also the “right shift” ν defined as

$$\nu(i) = \begin{cases} i+1 & \text{for } 1 \leq i < n, \\ \text{is not defined} & \text{for } i = n. \end{cases}$$

Theorem 1.8 (Solomon). *The semigroup R_n is generated by the elements $\sigma_1, \dots, \sigma_{n-1}, \nu$ with the following relations:*

- (1) *the Coxeter relations for the group S_n ;*
- (2) $\nu^{i+1} \sigma_i = \nu^{i+1}$;
- (3) $\sigma_i \nu^{n-i+1} = \nu^{n-i+1}$;
- (4) $\sigma_i \nu = \nu \sigma_{i+1}$;
- (5) $\nu \sigma_1 \sigma_2 \sigma_3 \cdots \sigma_{n-1} \nu = \nu$,

where $1 \leq i \leq n-1$ in (1)–(3) and (5), and $1 \leq i \leq n-2$ in (4).

2. The representation theory of the infinite symmetric inverse semigroup

In this section we assume that the reader is familiar with the basic notions and results of the theory of locally semisimple and AF algebras. Besides, we use some facts from the representation theory of the finite symmetric groups S_n and the infinite symmetric group S_∞ . See, e.g., [20].

There is a natural embedding $R_n \subset R_{n+1}$ of semigroups under which every map from R_n goes to a map from R_{n+1} that sends the element $n+1$ to itself. Consider the inductive limit of the chain $R_0 \subset R_1 \subset \cdots \subset R_n \subset \cdots$ of semigroups, which we will call the *infinite symmetric inverse semigroup* R_∞ .

2.1. The branching graph of the algebra $\mathbb{C}[R_\infty]$. Let \mathbb{Y} be the Young graph, and let \mathbb{Y}_n be the level of \mathbb{Y} whose vertices are indexed by all partitions of the integer n (Young diagrams with n cells). By $|\lambda|$ we denote the number of cells in a diagram λ (the sum of the parts of the partition λ).

Denote by $\tilde{\mathbb{Y}}$ the branching graph of the semigroup algebra $\mathbb{C}[R_\infty]$. It was described by Halverson [12].

Theorem 2.1 (Halverson). *The branching graph $\tilde{\mathbb{Y}}$ can be described as follows:*

- (1) *the vertices of the n th level are indexed by all Young diagrams with at most n cells: $\tilde{\mathbb{Y}}_n = \bigcup_{i=0}^n \mathbb{Y}_i$;*
- (2) *vertices $\lambda \in \tilde{\mathbb{Y}}_n$ and $\mu \in \tilde{\mathbb{Y}}_{n+1}$ are joined by an edge if either $\lambda = \mu$ or μ is obtained from λ by adding one cell.*

This leads us to the following definition of the *slow graph* $\tilde{\Gamma}$ constructed from a branching graph Γ :

(1) the set of vertices of the n th level of $\tilde{\Gamma}$ is the union of the sets of vertices of all levels of the original graph Γ with indices at most n , i.e., $\tilde{\Gamma}_n = \bigcup_{i=0}^n \Gamma_i$;

(2) vertices $\lambda \in \tilde{\Gamma}_n$ and $\mu \in \tilde{\Gamma}_{n+1}$ are joined by an edge if either $\lambda = \mu$ or μ is joined by an edge with λ in the original graph.

Recall the definition of the Pascal graph \mathbb{P} :

(1) the set \mathbb{P}_n of vertices of the n th level consists of all pairs of integers (n, k) , $0 \leq k \leq n$;

(2) vertices $(n, k) \in \mathbb{P}_n$ and $(n+1, l) \in \mathbb{P}_{n+1}$ are joined by an edge if either $l = k$ or $l = k+1$.

Observe that if the original graph Γ is the chain (the graph whose each level consists of a single vertex), then the corresponding slow graph $\tilde{\Gamma}$ coincides with \mathbb{P} . By analogy with the Pascal graph, we index the vertices of the n th level $\tilde{\Gamma}_n$ of the slow graph with the pairs (n, λ) , where $\lambda \in \Gamma_i$, $i \leq n$.

Remark 2.2. Note that if $G = \mathbb{P}$ is the Pascal graph, then the corresponding slow graph \tilde{G} is the three-dimensional analog of the Pascal graph. For the three-dimensional Pascal graph, the slow graph is the four-dimensional Pascal graph, etc. For the definition of the multidimensional analogs of the Pascal graph and a description of the traces of the corresponding algebras, see, e.g., [6].

Remark 2.3. The set of paths on the branching graph $\tilde{\mathbb{Y}}$ is in bijection with the random walks on \mathbb{Y} of the following form: at each moment, we are allowed either to stay at the same vertex or to descend to the previous level in an admissible way. In view of this description, graphs similar to $\tilde{\mathbb{Y}}$ are called slow.

Remark 2.4. In [7], the representation theory of the infinite Brauer algebra was studied. As in the previous remark, one can construct a bijection between the paths on the branching graph of the Brauer algebra and the random walks of a similar form on the Young graph: starting from the empty diagram, at each step we can move either to a vertex of the next level (joined by an edge with the current vertex) or to a vertex of the previous level (joined by an edge with the current vertex).

2.2. Facts from the theory of locally semisimple algebras. Given a branching graph Γ , denote by $T(\Gamma)$ the space of paths of Γ . On $T(\Gamma)$ we have the “tail” equivalence relation (see [4]): paths $x, y \in T(\Gamma)$ are equivalent, $x \sim y$, if they coincide from some level on. The partition of $T(\Gamma)$ into the equivalence classes will be denoted by $\xi = \xi_\Gamma$. Also, for every $k \in \mathbb{N} \cup 0$ and every path $s = (s_0, s_1, \dots, s_k)$ of length k , denote by $F_s \subset T(\Gamma)$ the cylinder $F_s = \{t \in T \mid t_i = s_i \text{ for } 0 \leq i \leq k\}$.

Given $x, y \in \Gamma$, by $\dim(x; y)$ denote the number of paths leading from x to y . By $\dim(y) = \dim(\emptyset; y)$ denote the total number of paths leading to y . By $\mathcal{E}(\Gamma)$ denote the set of ergodic central measures on $T(\Gamma)$. Given $\mu \in \mathcal{E}(\Gamma)$ and a vertex y , by $\mu(y)$ denote the measure of the set of all paths passing through y , i.e., the total measure of all cylinders F_s , $s = (s_0, s_1, \dots, s_{|y|})$, $s_{|y|} = y$.

We will use the following description of the characters of a locally semisimple algebra and the central measures on its branching graph (ergodic method).

Theorem 2.5 ([4]). *For every central ergodic measure μ , the set of paths $s = (s_0, s_1, \dots, s_f, \dots)$ such that*

$$\mu(y) = \lim_{f \rightarrow \infty} \frac{\dim(y) \cdot \dim(y; s_f)}{\dim s_f}$$

for all vertices y is of full measure.

Theorem 2.6 ([4]). *For every character ϕ of the algebra $A = C^*(\bigcup_{f=0}^{\infty} A_f)$, there exists a path $\{\lambda_f\}_{f=0}^{\infty}$ in the Bratteli diagram such that*

$$\phi(a) = \lim_{f \rightarrow \infty} \frac{\chi_{\lambda_f}(a)}{\dim \lambda_f}$$

for all $a \in A$. Here χ_{λ_f} is the character of the representation λ_f of the algebra A_f and $\dim \lambda_f$ is its dimension.

2.3. Description of the central measures on slow graphs. The key property of an arbitrary slow graph $\tilde{\Gamma}$ is that we can present the space of paths $T(\tilde{\Gamma})$ as the direct product of the spaces of paths $T(\Gamma)$ and $T(\mathbb{P})$. The same is true for the sets of paths between any two vertices. Moreover, the partition $\xi_{\tilde{\Gamma}}$ and the central ergodic measures on $T(\tilde{\Gamma})$ can also be presented as corresponding products.

Lemma 2.7. *Let Γ be the branching graph of a locally semisimple algebra and $\tilde{\Gamma}$ be the corresponding slow graph. Then*

1. $T(\tilde{\Gamma}) = T(\Gamma) \times T(\mathbb{P})$. Moreover, the number of paths between any two vertices of the slow graph $\tilde{\Gamma}$ is the product of the number of paths between the corresponding vertices of the original graph Γ and the number of vertices between the corresponding vertices of the Pascal graph \mathbb{P} :

$$\dim_{\tilde{\Gamma}}((n_1, \lambda_1); (n_2, \lambda_2)) = \dim_{\Gamma}(\lambda_1, \lambda_2) \cdot \dim_{\mathbb{P}}((n_1, |\lambda_1|); (n_2, |\lambda_2|)). \quad (1)$$

2. Let $s_{\tilde{\Gamma}}, t_{\tilde{\Gamma}} \in T(\tilde{\Gamma})$, $s_{\Gamma}, t_{\Gamma} \in T(\Gamma)$, $s_{\mathbb{P}}, t_{\mathbb{P}} \in T(\mathbb{P})$, and let $s_{\tilde{\Gamma}}$ correspond to the pair $(s_{\Gamma}, s_{\mathbb{P}})$ and $t_{\tilde{\Gamma}}$ correspond to the pair $(t_{\Gamma}, t_{\mathbb{P}})$. Then $s_{\tilde{\Gamma}} \sim t_{\tilde{\Gamma}}$ (with respect to $\xi_{\tilde{\Gamma}}$) if and only if $s_{\Gamma} \sim t_{\Gamma}$ (with respect to ξ_{Γ}) and $s_{\mathbb{P}} \sim t_{\mathbb{P}}$ (with respect to $\xi_{\mathbb{P}}$).

PROOF. 1. To each path in the graph $\tilde{\Gamma}$ there corresponds a unique strictly increasing sequence of vertices of the original graph Γ . Moreover, to each path $(i, \lambda_i)_{i=n_1}^{n_2}$ in the graph $\tilde{\Gamma}$ we can associate the path $(i, |\lambda_i|)_{i=n_1}^{n_2}$ in the Pascal graph. It is easy to see that the original path is uniquely determined by the constructed pair of paths, whence $T(\tilde{\Gamma}) = T(\Gamma) \times T(\mathbb{P})$.

Note that the constructed map determines a bijection between the paths from a vertex (n_1, λ_1) to a vertex (n_2, λ_2) in the graph $\tilde{\Gamma}$ and the pairs of paths between the corresponding vertices in the original graph Γ and in the Pascal graph \mathbb{P} , which proves formula (1).

2. The bijection in the proof of Claim 1 is constructed in such a way that the tail of a path $t_{\tilde{\Gamma}} = (t_{\Gamma}, t_{\mathbb{P}})$ depends only on the tails of the paths t_{Γ} and $t_{\mathbb{P}}$, and vice versa. \square \square

Theorem 2.8 (Description of the central measures). *There is a natural bijection $\mathcal{E}(\tilde{\Gamma}) \cong \mathcal{E}(\Gamma) \times \mathcal{E}(\mathbb{P})$. Every central ergodic measure $M_{\tilde{\Gamma}} \in \mathcal{E}(\tilde{\Gamma})$ is the product of central ergodic measures $M_{\Gamma} \in \mathcal{E}(\Gamma)$ and $M_{\mathbb{P}} \in \mathcal{E}(\mathbb{P})$; namely, $M_{\tilde{\Gamma}}(F_{(n, \lambda)}) = M_{\Gamma}(F_{\lambda}) \cdot M_{\mathbb{P}}(F_{(n, |\lambda|)})$ for every cylinder $F_{(n, \lambda)}$.*

PROOF. In accordance with the decomposition $T(\tilde{\Gamma}) = T(\Gamma) \times T(\mathbb{P})$, given a central ergodic measure $M_{\tilde{\Gamma}} \in \mathcal{E}(\tilde{\Gamma})$, consider the projections $M_{\Gamma} \in \mathcal{E}(\Gamma)$ and $M_{\mathbb{P}} \in \mathcal{E}(\mathbb{P})$ defined as follows:

$$M_{\Gamma}(F_{\lambda}) = \sum_{n \geq |\lambda|} M_{\tilde{\Gamma}}(F_{(n, \lambda)}), \quad M_{\mathbb{P}}(F_{(n, k)}) = \sum_{|\lambda|=k} M_{\tilde{\Gamma}}(F_{(n, \lambda)}).$$

The measures M_{Γ} and $M_{\mathbb{P}}$ are central by the centrality of $M_{\tilde{\Gamma}}$.

Further, according to formula (1) from Lemma 2.7 and Theorem 2.5,

$$\begin{aligned}
M_{\tilde{\Gamma}}(F_{(n,\lambda)}) &= \lim_{f \rightarrow \infty} \frac{\dim((n, \lambda_n); (f, \lambda_f))}{\dim(f, \lambda_f)} \\
&= \lim_{f \rightarrow \infty} \frac{\dim_{\mathbb{P}}((n, |\lambda_n|); (f, |\lambda_f|))}{\dim_{\mathbb{P}}(f, |\lambda_f|)} \cdot \frac{\dim_{\Gamma}(\lambda_n; \lambda_f)}{\dim_{\Gamma}(\lambda_f)} \\
&= \lim_{f \rightarrow \infty} \frac{\dim_{\mathbb{P}}(n, |\lambda_n|); (f, |\lambda_f|))}{\dim_{\mathbb{P}}(f, |\lambda_f|)} \cdot \lim_{f \rightarrow \infty} \frac{\dim_{\Gamma}(\lambda_n; \lambda_f)}{\dim_{\Gamma}(\lambda_f)}. \quad (2)
\end{aligned}$$

The limits in the right-hand side of (2) exist and are equal to $M_{\Gamma}(F_{\lambda})$ and $M_{\mathbb{P}}(F_{(n,k)})$, which proves the required formula for $M_{\tilde{\Gamma}}$. The ergodicity of the measures M_{Γ} and $M_{\mathbb{P}}$ follows from the ergodicity of the measure $M_{\tilde{\Gamma}}$.

Conversely, the product (in the above sense) of central ergodic measures $M_{\Gamma} \in \mathcal{E}(\Gamma)$ and $M_{\mathbb{P}} \in \mathcal{E}(\mathbb{P})$ is a central ergodic measure $M_{\tilde{\Gamma}} \in \mathcal{E}(\tilde{\Gamma})$. Its centrality follows from Lemma 2.7, and its ergodicity follows from equation (2). \square \square

Recall (see, e.g., [6]) that for the Pascal graph \mathbb{P} , the limits in Theorem 2.5 exist if and only if for the path

$$((0, k_0), (1, k_1), \dots, (f, k_f), \dots)$$

the limit

$$\lim_{f \rightarrow \infty} k_f/f = \delta, \quad \delta \in [0; 1], \quad (3)$$

does exist, and to every $\delta \in [0; 1]$ there corresponds a unique central measure $M_{\mathbb{P}} = M_{\mathbb{P}}^{\delta}$.

Corollary 2.9. *Every measure $M_{\tilde{\Gamma}} \in \mathcal{E}(\tilde{\Gamma})$ is parameterized by a pair (δ, M_{Γ}) , $\delta \in [0; 1]$, $M_{\Gamma} \in \mathcal{E}(\Gamma)$.*

Corollary 2.10. *The measure $M_{\tilde{\Gamma}} = (\delta, M_{\Gamma})$ on $T(\tilde{\Gamma})$ is concentrated on paths for which the corresponding paths in the graph Γ lie in the support of the measure M_{Γ} and, besides, the limit (3) does exist.*

In particular, consider an arbitrary central ergodic measure $M_{\mathbb{Y}}$ on the graph \mathbb{Y} corresponding to parameters $\alpha = \{\alpha_i\}$, $\beta = \{\beta_i\}$, γ . Then the measure $M_{\tilde{\mathbb{Y}}} = (\delta, M_{\mathbb{Y}})$ on $T(\tilde{\mathbb{Y}})$ is concentrated on paths of the form $\{(f, \lambda_f)\}$ for which the corresponding limits for the sequence $\{\lambda_f\}$ are equal to $\{\alpha_i\}$ and $\{\beta_i\}$ and, besides, $\lim_{f \rightarrow \infty} |\lambda_f|/f = \delta$.

2.4. A formula for the characters of the infinite symmetric semi-group. The bijection described above between the sets of central measures on the spaces of paths of the graph Γ and of the slow graph $\tilde{\Gamma}$ holds for an arbitrary graded graph Γ . This bijection can be translated to the sets of characters of the algebras corresponding to these graphs (see Corollary 2.11 below) via the correspondence between central measures and characters; however, explicit formulas for characters substantially depend on the graphs and algebras and have no universal meaning. Below we prove a formula that expresses a character of the algebra $\mathbb{C}[R_{\infty}]$ in terms of the corresponding character of the algebra $\mathbb{C}[S_{\infty}]$. In this section, by a character we always mean an indecomposable character.

Corollary 2.11. *The parametrization of the set of central measures described above determines a bijection which sends every pair $(\delta, \chi_{\alpha, \beta, \gamma}^{S_{\infty}})$, where $\delta \in [0, 1]$ and $\chi_{\alpha, \beta, \gamma}^{S_{\infty}}$ is a character of the algebra $\mathbb{C}[S_{\infty}]$, to the character $\chi_{\alpha, \beta, \gamma, \delta}^{R_{\infty}}$ of the algebra $\mathbb{C}[R_{\infty}]$.*

To simplify the notation, below we often omit the superscripts and the parameter γ (which can be expressed in terms of α and β), setting

$$\chi_{\alpha,\beta} \equiv \chi_{\alpha,\beta,\gamma}^{S_\infty}, \quad \chi_{\alpha,\beta,\delta} \equiv \chi_{\alpha,\beta,\gamma,\delta}^{R_\infty}.$$

The conjugation of an element $\sigma \in R_n$ by an element of the symmetric group does not change the value of a character, so it suffices to consider *reduced elements* $\sigma^\circ \in R_n$, for which all fixed points are at the end: for every $\sigma \in R_n$ there exist $g \in S_n$, $n(\sigma) \in \mathbb{N} \cup 0$ such that $\sigma^\circ = g\sigma g^{-1}$ and $\sigma^\circ(i) \neq i$ for $i < n(\sigma)$ and $\sigma^\circ(i) = i$ for $i \geq n(\sigma)$. By the definition of the embedding $R_n \subset R_{n+1}$, we may assume that $\sigma^\circ \in R_{n(\sigma)}$. The order $n(\sigma)$ of the element σ° is uniquely determined by the element σ .

Let us introduce a set $M_k(\sigma) \subset S_n$ whose elements are indexed by all k -element subsets $K \subset \{1, \dots, n\}$ fixed under σ : to each such subset we associate the bijection $\tilde{\sigma} \in S_n$ that coincides with σ on K and is identity at all other points.

Note that for every element σ of the semigroup R_n we may consider the maximal (possibly, empty) subset of $\{1, \dots, n\}$ that is mapped by σ to itself in a one-to-one manner. The restriction of σ to this subset will be called the *invertible part* of σ . The invertible part of every element $\sigma \in R_n$ can be regarded as an element of some symmetric group S_r , $r \leq n$, and, consequently, it can be written as a product of disjoint cycles. The set $M_k(\sigma)$ can also be parameterized by the set of all subcollections of cycles of total length k from the cycle decomposition of the invertible part of σ .

In the next theorem, the value of an indecomposable character of the infinite symmetric semigroup at an element $\sigma \in R_n$ is presented as a linear combination of the values of the corresponding Thoma character at each of the elements of the disjoint union $\bigsqcup_k M_k(\sigma)$ with coefficients depending only on the parameter δ .

Theorem 2.12 (A formula for the characters). *Let $\chi_{\alpha,\beta,\gamma,\delta}^{R_\infty} \equiv \chi_{\alpha,\beta,\delta}$ be an indecomposable character of the algebra $\mathbb{C}[R_\infty]$, $\chi_{\alpha,\gamma,\beta}^{S_\infty} \equiv \chi_{\alpha,\beta}$ be the corresponding indecomposable character of the algebra $\mathbb{C}[S_\infty]$, and $\sigma \in R_\infty$ be a reduced element. Then*

$$\chi_{\alpha,\beta,\delta}(\sigma) = \sum_{k=0}^{n\sigma} \left(\delta^{n(\sigma)-k} (1-\delta)^k \cdot \sum_{\tilde{\sigma} \in M_k(\sigma)} \chi_{\alpha,\beta}(\tilde{\sigma}) \right).$$

PROOF. By Theorem 2.6, there exists a path $\{(f, \lambda_f)\}_{f=0}^\infty$ such that

$$\chi_{\alpha,\beta,\delta}(\sigma) = \lim_{f \rightarrow \infty} \frac{\chi_{(f, \lambda_f)}^*(\sigma)}{\dim(f, \lambda_f)}.$$

Recall that an element $\sigma \in R_n$ is regarded as an element of the semigroup R_f that is identity on the subset $\{n+1, \dots, f\}$. By Theorem 1.5, in order to compute the character $\chi_{(f, \lambda_f)}^*(\sigma)$, it suffices to describe subsets of size $|\lambda_f|$ in the set $\{1, \dots, f\}$ fixed under the action of the element $\sigma \in R_f$. In order to completely describe such subsets, it suffices to associate with every fixed subset of size k in the set $\{1, \dots, n\}$ all possible subsets of $|\lambda_f| - k$ fixed points in the set $\{n+1, \dots, f\}$. Thus

$$\chi_{(f, \lambda_f)}^*(\sigma) = \sum_k \left(\binom{f-n}{|\lambda_f| - k} \cdot \sum_{\tilde{\sigma} \in M_k(\sigma)} \chi_{\lambda_f}(\tilde{\sigma}) \right).$$

By Claim 1 of Lemma 2.7,

$$\begin{aligned}\chi_{\alpha,\beta,\delta}(\sigma) &= \lim_{f \rightarrow \infty} \frac{\sum_k \left(\binom{f-n}{|\lambda_f|-k} \cdot \sum_{\tilde{\sigma}} \chi_{\lambda_f}(\tilde{\sigma}) \right)}{\dim(f, |\lambda_f|) \cdot \dim(\lambda_f)} \\ &= \sum_k \left(\lim_{f \rightarrow \infty} \frac{\binom{f-n}{|\lambda_f|-k}}{\dim(f, |\lambda_f|)} \cdot \sum_{\tilde{\sigma}} \lim_{f \rightarrow \infty} \frac{\chi_{\lambda_f}(\tilde{\sigma})}{\dim(\lambda_f)} \right).\end{aligned}\quad (4)$$

According to Corollary 2.10 and Theorem 2.6 applied to the infinite symmetric group S_∞ , each of the summands in the right factor in the right-hand side of (4) tends to the corresponding value of the character $\chi_{\alpha,\beta}$. Besides, by Corollary 2.10, $\lim |\lambda_f|/f = \delta$, whence

$$\lim_{f \rightarrow \infty} \frac{\binom{f-n}{|\lambda_f|-k}}{\dim(f, |\lambda_f|)} = \delta^{n-k}(1-\delta)^k,$$

and this completes the proof. \square

Corollary 2.13. *For an arbitrary element $\sigma \in R_n \subset R_\infty$,*

$$\chi_{\alpha,\beta,\delta}(\sigma) = \sum_{k=0}^n \left(\delta^{n-k}(1-\delta)^k \cdot \sum_{\tilde{\sigma} \in M_k(\sigma)} \chi_{\alpha,\beta}(\tilde{\sigma}) \right).$$

Corollary 2.14. *The restriction of a character $\chi_{\alpha,\beta,\delta}$ of the algebra $\mathbb{C}(R_\infty)$ to $\mathbb{C}(S_\infty)$ is equal to $\chi_{\alpha',\beta'}$, where $\alpha'_1 = \delta$, $\alpha'_i = (1-\delta)\alpha_{i-1}$ for $i > 1$ and $\beta' = (1-\delta)\beta$.*

PROOF. We will verify the assertion in the case $\beta = 0$. Let $\alpha'_1 = \delta$, $\alpha'_i = (1-\delta)\alpha_{i-1}$ for $i > 1$, and $\sigma \in S_n$. Then

$$\chi_{\alpha',0}^{S_\infty}(\sigma) = \prod_{\gamma} \left((1-\delta)^{k_\gamma} \cdot \sum_i \alpha_i^{k_\gamma} + \delta^{k_\gamma} \right),$$

where the product is taken over all minimal cycles γ in the cycle decomposition of the element σ and k_γ are the lengths of these cycles. Expanding the product, we obtain

$$\chi_{\alpha',0}^{S_\infty}(\sigma) = \sum_k \sum_{\tilde{\sigma} \in M_k(\sigma)} \left((1-\delta)^k \delta^{n-k} \cdot \prod_{\gamma} \left(\sum_i \alpha_i^{k_\gamma} \right) \right),$$

where the internal product is taken over all minimal cycles γ of the subcollection $\tilde{\sigma}$. Writing the last equation in the form

$$\chi_{\alpha',0}^{S_\infty}(\sigma) = \sum_k \left(\delta^{n-k}(1-\delta)^k \cdot \sum_{\tilde{\sigma} \in M_k(\sigma)} \chi_{\alpha,0}^{S_\infty}(\tilde{\sigma}) \right) = \chi_{\alpha',0,\delta}^{R_\infty}(\sigma),$$

we obtain the desired assertion. \square

Remark 2.15. In the previous corollary, the parameters α and β are not symmetric, despite the fact that in the graph \mathbb{Y} the symmetry is present. The reason is as follows: under the embedding of the group S_n into the semigroup R_n , the restriction of an irreducible representation of R_n to S_n is the representation induced from a representation of the subgroup $S_r \times S_{n-r} \subset S_n$ that is trivial on the second factor, see Remark 1.3. Hence the operation of restricting a representation does not commute with the involution (see Remark 1.4), which breaks the symmetry between the parameters α and β .

2.5. Realization of representations. We turn our attention to the case where $\sum_i \alpha_i = 1$, i.e., $\beta_i = 0$ for all i . Consider a measure on \mathbb{N} of the form $\mu_\alpha(i) = \alpha_i$, the set of sequences $\mathcal{X} = \prod \mathbb{N}$ equipped with the measure $m_\alpha = \prod \mu_\alpha$, and the set $\tilde{\mathcal{X}}$ of pairs of sequences coinciding from some point on. In the space $L^2(\tilde{\mathcal{X}}, m_\alpha)$ we can realize the representation of the symmetric group S_∞ corresponding to the Thoma parameters $(\alpha, 0)$, see [5], [21].

Theorem 2.16. *The realization of the representation of the group S_∞ corresponding to the parameters $(\alpha', 0)$, where α' is defined in Corollary 2.14, in the space of functions $L^2(\tilde{\mathcal{X}}, m_{\alpha'})$ can be extended to a realization of the representation of the semigroup R_∞ corresponding to the parameters $(\alpha, 0, \delta)$.*

PROOF. Define the action of the projection p_1 from Theorem 1.6 as follows: it maps every sequence $(a_1, a_2, a_3, \dots) \in \mathcal{X}$ to the sequence $(1, a_2, a_3, \dots) \in \mathcal{X}$. The relations from Theorem 1.6 are obviously satisfied.

Thus it suffices to check that introducing an additional projection does not lead beyond the space of the representation. But, as shown in [3], the space of the factor representation of the symmetric group S_∞ coincides with the whole space $L^2(\tilde{\mathcal{X}}, m_{\alpha'})$, which completes the proof. \square \square

Corollary 2.17. *In terms of the realization described above, one can give a short formula for the characters of R_∞ , similar to the formula for the characters of the symmetric group (cf. [5]), which expresses the value of a character at an element σ as the measure of the set of fixed points of σ ; namely,*

$$\chi_{\alpha, 0, \delta}(\sigma) = m_{\alpha'}(\{x : \sigma(x) = x\}),$$

where α' is defined in Corollary 2.14. See also [3].

3. Appendix. General information on finite inverse semigroups

In this section, we mainly follow the monograph [9] and the paper [2].

3.1. The definition of an inverse semigroup.

Theorem 3.1. *The following two conditions on a semigroup S are equivalent:*

- (1) *for every $a \in S$ there exists $x \in S$ such that $axa = a$, and any two idempotents of S commute;*
- (2) *every principal left ideal and every principal right ideal of S is generated by a unique idempotent;*
- (3) *for every $a \in S$ there exists a unique $x \in S$ such that $axa = a$ and $xax = x$.*

A semigroup satisfying the conditions of Theorem 3.1 is called an *inverse semigroup*. One says that the elements a and x from condition (1) of the theorem are *inverse* to each other; sometimes, this is denoted as $x = a^{-1}$. Note that $(ab)^{-1} = b^{-1}a^{-1}$ for any $a, b \in S$.

Let us prove that the symmetric inverse semigroup is an inverse semigroup. Given a partial map $\sigma \in R_n$ that acts from a subset $X \subset \{1, \dots, n\}$ to a subset $Y \subset \{1, \dots, n\}$, we construct the map σ^{-1} from Y to X inverse to σ in the ordinary sense, i.e., for $y \in Y$ and $x \in X$ we set $\sigma^{-1}(y) = x$ if $\sigma(x) = y$. The elements σ and σ^{-1} are obviously inverse to each other. Besides, the idempotents of the symmetric inverse semigroup are exactly those maps that send some subset $X \subset \{1, \dots, n\}$ to itself and are not defined on $\{1, \dots, n\} \setminus X$. Therefore, any two idempotents commute, and the semigroup is inverse by Theorem 3.1.

3.2. An analog of Cayley's theorem. Vagner [1] and Preston [16] proved for inverse semigroups an analog of Cayley's theorem for groups.

Theorem 3.2. *An arbitrary inverse semigroup S is isomorphic to an inverse sub-semigroup of the symmetric inverse semigroup of all one-to-one partial transformations of the set S .*

The proof is much more difficult than in the group case, and we do not reproduce it (see [9]). Note that the theorem holds both for finite and infinite inverse semigroups.

3.3. The semisimplicity of the semigroup algebra. Given an arbitrary finite semigroup S and a field F , one can consider the semigroup algebra $F[S]$ of S over F . The elements of S form a basis in $F[S]$, and the multiplication law for these basis elements coincides with the multiplication law in S . Necessary and sufficient conditions for the semisimplicity of the semigroup algebra $F[S]$ of a finite inverse semigroup S were obtained independently by Munn [14] and Oganesyan [10].

Theorem 3.3. *The semigroup algebra $F[S]$ of a finite inverse semigroup S over a field K is semisimple if and only if the characteristic of K is zero or a prime that does not divide the order of any subgroup in S .*

3.4. Involutive bialgebras and semigroup algebras of inverse semigroups. A *bialgebra* (see [8]) is a vector space over the field \mathbb{C} equipped with compatible structures of a unital associative algebra and a counital coassociative coalgebra. Namely, the following equivalent conditions are satisfied:

- (1) the comultiplication and the counit are homomorphisms of the corresponding algebras;
- (2) the multiplication and the unit are homomorphisms of the corresponding coalgebras.

Let us also introduce the notion of a *weakened bialgebra* for the case where the multiplication and comultiplication are homomorphisms, but there is no condition on the unit and counit.

The group algebra of a finite group with the convolution multiplication and diagonal comultiplication is a cocommutative bialgebra (and even a Hopf algebra). It is well known (see [8]) that the semigroup algebra of every finite semigroup with identity (monoid) is also a cocommutative bialgebra with the natural definition of the operations.

An involution of an algebra is a second-order antilinear antiautomorphism of this algebra; a second-order antilinear antiautomorphism of a coalgebra is called a coinvolution. A bialgebra equipped with an involution and a coinvolution is called an *involutive bialgebra*, or a bialgebra with involution, if the multiplication commutes with the coinvolution and the comultiplication commutes with the involution.

In [2] it was shown that the class of finite inverse semigroups generates exactly the class of involutive semisimple bialgebras.

Theorem 3.4. *The semigroup algebra of a finite inverse semigroup is a semisimple cocommutative involutive algebra. Analogously, the dual semigroup algebra $\mathbb{C}[S]$ of a finite inverse semigroup S with identity is a commutative involutive bialgebra. Conversely, every finite-dimensional semisimple cocommutative (in the dual case, commutative) involutive bialgebra is isomorphic (as an involutive bialgebra)*

to the semigroup algebra (respectively, dual semigroup algebra) of a finite inverse semigroup with identity.

For inverse semigroups without identity, the semigroup bialgebra is a weakened bialgebra (the counit is not a homomorphism).

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